Spring 2022 Math 290: Solutions to Exam 1

The total value of this exam is 100 points. Be sure to check your calculations carefully. Do not write your solutions on the exam sheet. At 10:50 stop working, copy your solutions and upload a pdf copy to Canvas. **No solutions will be accepted after 11am**. Make sure your name is on the first page of your uploaded solution set.

No calculators, phones or laptops may be used during this exam.

1. True-False and short answer questions. (5 points each)

(i) Explain what it means for an $m \times n$ matrix A to have rank r.

(ii) True or false: If A and B are invertible $n \times n$ matrices. Then AB is invertible. If true, give a formula for the inverse of AB; if false, explain why.

(iii) True or False: Every $m \times n$ matrix can be put into reduced row echelon form by a sequence of elementary row operations.

(iv) Write the following system of equations as a matrix equation:

$$\begin{aligned} 2x + 3y - 4z &= 7 \\ -6y + \sqrt{2}z &= \pi \\ x - y + z &= 10. \end{aligned}$$

(v) Write down the inverse of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$.

Solution. (i) The rank of A is the number of leading ones in the reduced row echelon form of A. Alternately: If A is the coefficient matrix of a system of linear equations, the rank of A is n minus the number of independent parameters needed to describe the solution set.

(ii) True. $(AB)^{-1} = B^{-1}A^{-1}$.

(iii) True.

(iv) $\begin{bmatrix} 2 & 3 & -4 \\ 0 & -6 & \sqrt{2} \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ \pi \\ 10 \end{bmatrix}$.

(v) The given matrix corresponds to the row operation $7 \cdot R_2 + R_3$. The inverse row operation is $-7 \cdot R_2 + R_3$. Thus, the inverse matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 1 \end{bmatrix}$.

2. Solve the homogeneous system of equations below by using Gaussian elimination and then write your answer as a column expressed in terms of the basic solutions, using the parameters s and t. Identify the basic solutions. (35 points)

$$\begin{aligned} x + 4y - 3w &= 0 \\ -2y + z + 6w &= 0 \\ 2x + 8y - 6w &= 0 \\ x + 2y + z + 3w &= 0. \end{aligned}$$

Solution.

$$\left[\begin{array}{cccc|c} 1 & 4 & 0 & -3 & 0 \\ 0 & -2 & 1 & 6 & 0 \\ 2 & 8 & 0 & -6 & 0 \\ 1 & 2 & 1 & 3 & 0 \end{array}\right] \xrightarrow[-1 \cdot R_1 + R_4]{-2 \cdot R_1 + R_3} \left[\begin{array}{cccc|c} 1 & 4 & 0 & -3 & 0 \\ 0 & -2 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 6 & 0 \end{array}\right] \xrightarrow{-1 \cdot R_2 + R_4} \left[\begin{array}{cccc|c} 1 & 4 & 0 & -3 & 0 \\ 0 & -2 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}\cdot R_2} \left[\begin{array}{cccc|c} 1 & 4 & 0 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-4\cdot R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 9 & 0 \\ 0 & 1 & -\frac{1}{2} & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Writing the solution as a column, we have

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2s-9t \\ \frac{1}{2}s+3t \\ s \\ t \end{bmatrix} = s\cdot\begin{bmatrix} -2 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t\cdot\begin{bmatrix} -9 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, two basic solutions are $\begin{bmatrix} -2 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -9 \\ 3 \\ 0 \\ 1 \end{bmatrix}$.

3. If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, Calculate $(B\cdot A)^t + 2C^2$. (10 points)

Solution. $(B\cdot A)^t = \begin{bmatrix} 0 & 6 \\ 12 & 5 \end{bmatrix}^t = \begin{bmatrix} 0 & 12 \\ 6 & 5 \end{bmatrix}$. $2c^2 = 2\begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 12 & 14 \end{bmatrix}$. $(B\cdot A)^t + 2C^2 = \begin{bmatrix} 14 & 20 \\ 18 & 19 \end{bmatrix}$.

4. Suppose we are given a 3×5 matrix A, and we consecutively perform the following elementary row operations on A to obtain the new matrix B.

(i) $-3\cdot R_3 + R_1$
(ii) $2\cdot R_1$
(iii) $R_1 \leftrightarrow R_2$.

Find C such that $CA = B$. (15 points)

Solution. There are two ways to solve this problem. The easy way is to realize that the matrix C is the result of performing the indicated row operation on the identity matrix. This gives:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3\cdot R_3+R_1} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2\cdot R_1} \begin{bmatrix} 2 & 0 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & -6 \\ 0 & 0 & 1 \end{bmatrix}.$$

The more standard way is to multiply the corresponding elementary matrices:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & -6 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Given $A = \begin{bmatrix} 0 & 1 \\ 3 & 6 \end{bmatrix}$, write A and A^{-1} as a product of elementary matrices. DO NOT multiply these products out. (15 points)

Solution. Employing elementary row operations, we have:

$$\begin{bmatrix} 0 & 1 \\ 3 & 6 \end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix} 3 & 6 \\ 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{3}\cdot R_1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2\cdot R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In terms of elementary matrices this becomes:

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot A = I_2,$$

which shows that $A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Taking inverses of both sides, and keeping in mind that: (i) $(A^{-1})^{-1} = A$ and (ii) the inverse of a product of matrices is the product of the inverses in reverse order, we have $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.